A REPRESENTATION OF THE SOLUTION OF THE HEAT-CONDUCTION EQUATION

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A representation of the solution of the heat-conduction equation is examined in the form of a series, arranged in increasing order of derivatives of boundary time functions.

In some applications of heat-conduction theory use is made of a representation of the solution of the heat-conduction equation based on a series expansion in powers of the heat capacity of the medium [1, 2].

In the general case the solution of the one-dimensional heat conduction equation with coefficients depending on the coordinates may be represented by a series arranged in ascending order of derivatives of boundary time functions. In the case of a homogeneous medium, this series is also arranged in inverse powers of the thermal diffusivity. The representation in question, in particular, gives a convenient form of solution for problems relating to the propagation of heat in multilayer media, when the use of an operational method usually entails a large volume of calculation.

Let the heat-conduction equation be given in the interval $a < x < b$ as

$$\frac{\partial}{\partial x}\left[K(x)\frac{\partial T}{\partial x}\right] = c(x)\frac{\partial T}{\partial t} \tag{1}$$

with boundary conditions of the form

$$\alpha_1 \left.\frac{\partial T}{\partial x}\right|_{x=a} + \beta_1 T(a,t) = f(t),$$

$$\alpha_2 \left.\frac{\partial T}{\partial x}\right|_{x=b} + \beta_2 T(b,t) = g(t), \tag{2}$$

where one of the constants β_1 or β_2, β_1 for definiteness, is different from zero, or

$$K(a)\left.\frac{\partial T}{\partial x}\right|_{x=a} = f(t),\ K(b)\left.\frac{\partial T}{\partial x}\right|_{x=b} = g(t), \tag{2a}$$

and with an initial condition

$$T|_{t=0} = \omega(x). \tag{3}$$

We shall suppose that the boundary functions $f(t)$ and $q(t)$ are continuous in some region $0 \le t \le t_1$, and have derivatives of all orders:

$$\left|\frac{d^{(m)} f(t)}{dt^{(m)}}\right| \leqslant \nu_1 \left|\frac{d^{(m-1)} f(t)}{dt^{(m-1)}}\right|,$$

$$\left|\frac{d^{(m)} q(t)}{dt^{(m)}}\right| \leqslant \nu_2 \left|\frac{d^{(m-1)} q(t)}{dt^{(m-1)}}\right|,\ m = 1, 2, \ldots. \tag{4}$$

Integrating (1) twice with respect to x with $a \le x \le b$, and taking (2) into account, we obtain, from the continuity of the functions T, $K(x)(\partial T/\partial x)$ and $\partial T/\partial t$

$$T(x,t) = F(x,t) - \int_a^b Q(x,\xi)\frac{\partial T(\xi,t)}{\partial t}d\xi, \tag{5}$$

where

$$F(x,t) = \beta_1^{-1}[1 - \beta_2 A\psi(x)]f(t) + A\psi(x)g(t);$$

$$Q(x,\xi) = \begin{cases} \psi(\xi)[1 - \beta_2 A\psi(x)]c(\xi), & a \leqslant \xi \leqslant x \\ \psi(x)[1 - \beta_2 A\psi(\xi)]c(\xi), & x \leqslant \xi \leqslant b; \end{cases}$$

$$\psi(x) = \int_a^x \frac{dx}{K(x)} - \frac{\alpha_1}{\beta_1}\frac{1}{K(a)};\quad \frac{1}{A} = \frac{\alpha_2}{K(b)} + \beta_2\psi(b).$$

In the case of boundary conditions of type (2a), a result of a similar nature will be found for the function

$$q(x,t) = K(x)\frac{\partial T(x,t)}{\partial x}.$$

Relation (5) is an integral equation in the variable x and a differential equation with respect to the variable t. Its solution, constructed from the method of successive approximations, is

$$T(x,t) = F(x,t) + \sum_{n=1}^{\infty}(-1)^n T_n(x,t), \tag{6}$$

where

$$T_n(x,t) = \int_a^b Q_n(x,\xi)\frac{\partial^{(n)} F(\xi,t)}{\partial t^{(n)}}d\xi,$$

$$Q_{n+1}(x,\xi) = \int_a^b Q(x,\zeta)Q_n(\zeta,\xi)d\zeta,\ n = 1, 2, \ldots,$$

$$Q_1(x,\xi) = Q(x,\xi),$$

is a solution of the heat-conduction problem without the initial condition. It also gives a representation of the solution of the original problem (1)–(3) after the lapse of a certain time interval.

Solution (6) is determined to an accuracy up to an additive function w(x, t), satisfying the homogeneous equation

$$w(x,t) = -\int_a^b Q(x,\xi)\frac{\partial w(\xi,t)}{\partial t}d\xi. \tag{7}$$

By suitable choice of this function we may satisfy the initial condition (3). Then w(x, t) will have the form [3]

$$w(x,t)=\frac{1}{\sqrt{c(x)}}\times$$
$$\times\sum_{n=1}^{\infty}u_n(x)\exp(-\lambda_n t)\int_a^b\sqrt{c(x)}\,[\omega(x)-T(x,0)]\,u_n(x)\,dx, \quad (8)$$

where λ_n and $u_n(x)$ are characteristic values and normalized eigenfunctions of the homogeneous integral equation with symmetrical kernel

$$u(x)=\lambda\int_a^b\sqrt{c(x)/c(\xi)}\,Q(x,\xi)\,u(\xi)\,d\xi,$$

and T(x, 0) is the value of function (6) when t = 0.

It may be shown that an estimate exists for the quantity w(x, t)

$$|w(x,t)|\leqslant CM\exp\left(-\frac{t}{B'}\right)\int_a^b c(x)\,dx, \quad (9)$$

where

$$M=\max\left|\frac{1}{c(x)}\frac{\partial}{\partial x}\left\{K(x)\frac{\partial}{\partial x}[\omega(x)-T(x,0)]\right\}\right|,\ a\leqslant x\leqslant b,$$
$$B'^2=\int_a^b\int_a^b\frac{c(x)}{c(\xi)}Q^2(x,\xi)\,dx\,d\xi,$$
$$C=\max\int_a^b\frac{1}{c(\xi)}|Q(x,\xi)|\,d\xi,\ a\leqslant x\leqslant b.$$

The convergence of series (6) may be established from analogy with the convergence of the successive approximations in the theory of Fredholm equations [4]. Then a sufficient condition for convergence has the form

$$B\nu<1, \quad (10)$$

where

$$B^2=\int_a^b\int_a^b Q^2(x,\xi)\,dx\,d\xi;\ \nu=\max(\nu_1,\nu_2).$$

If we restrict series (6) to the first n terms, the resulting error is

$$R_n\leqslant\frac{B^n\nu^{n+1}\sqrt{C_1}\,(|L_1 f(t)|+|L_2 g(t)|)}{1-B\nu}, \quad (11)$$

where

$$C_1=\max\int_a^b Q^2(x,\xi)\,d\xi,\ a\leqslant x\leqslant b,$$
$$L_1^2=\beta_1^{-2}\int_a^b[1-\beta_2 A\psi(x)]^2\,dx,$$
$$L_2^2=A^2\int_a^b\psi^2(x)\,dx.$$

In the case of a homogeneous medium c(x) = c, K(x) = K, c/K = $1/\varkappa$, b − a = l, and the quantities T_n(x, t) and B may be represented in the form

$$T_n(x,t)=\frac{1}{\varkappa^n}\overline{T}_n(x,t),\ B=\frac{l^2}{\varkappa}\overline{B},$$

where the function $\overline{T}_n$(x, t) and the quantity $\overline{B}$ do not depend on the heat capacity, and so the expansion (6) coincides with the expansion of the temperature function in a series of inverse powers of the thermal diffusivity.

NOTATION

T–temperature; t–time; x, ξ–coordinates; K(x)–thermal conductivity; c(x)–volume heat capacity; f(t) and q(t)–boundary functions.

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